



Black holes with multiple horizons without mass inflation

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Abstract Mass inflation is a phenomenon occurring in the vicinity of the inner horizon in a spacetime with two horizons. It is shown that the mass of initially small perturbations grows exponentially as they approach the inner horizon. This implies that the inner horizon of a black hole is unstable under perturbations. Since mass inflation is determined by the surface gravities of the inner horizons, Carballo-Rubio et al. showed that if one makes the surface gravities of the inner horizons vanish, then the exponential growth associated with mass inflation is not present. Based on this conclusion, we look for the black hole solutions of multiple horizons in the presence of a nonlinear Maxwell field. We then make the inner horizons coincide with each other such that the surface gravities of all inner horizons vanish. In this way, black holes with multiple horizons without mass inflation are constructed.

1 Introduction

The concept of mass inflation describes an exponential growth of energy [1–3] in the sense that initially small infalling perturbations grow exponentially as they approach the inner horizon of a spacetime with two horizons. It is emphasized that the divergence of energy happens in the vicinity of the inner horizon. We know that multiple inner horizons can be present in multi-horizon black holes. Therefore, the mass inflation phenomenon may also occur in such black holes.

Then how can this problem be avoided? Carballo-Rubio et al. [4] showed that, if we make an attempt to let the surface gravities of the inner horizons vanish, then the exponen-

tial growth associated with mass inflation instability is not present. In practice, they constructed specific geometries for regular black holes without mass inflation and analyzed their behavior with the double null shell method and the modified Ori method, showing that the exponential growth of mass inflation disappears when the surface gravities of the inner horizons vanish. In this context, Eichhorn and Fernandes [5] have also recently derived regular black-hole solutions. Because these solutions have vanishing surface gravities, they are not susceptible to mass-inflation instability. Very recently, Lin et al. [6] explored whether mass inflation occurs in black-hole solutions obtained in Loop Quantum Gravity. They found that it does. Therefore, the mass inflation problem is far from being settled.

Actually, in terms of a suitable null coordinate v , the mass inflation is given by [7]

$$\delta m(v) \propto (v - v_0)^{-p} e^{|\kappa_{in}|(v-v_0)}, \quad (1)$$

where κ_{in} is the surface gravity of the inner horizon. It is obvious that mass inflation is determined by the surface gravity κ_{in} of the inner horizon. This suggests that if we let κ_{in} vanish, the mass inflation can be switched off.

How can the surface gravities of inner horizons be made to vanish? Note that when the two horizons of Reissner–Nordström spacetime coincide, the surface gravity of the resulting horizon is exactly zero and mass inflation is erased. Inspired by this fact, we seek black holes with multiple horizons. We then require that the surface gravities of all the inner horizons vanish by making the inner horizons coincide. We shall work within the framework of Einstein nonlinear-Maxwell theory.

Black holes with many horizons have mainly been studied in Refs. [8–10]. Specifically, by using the solution-generating method, Nojiri and Odintsov [8] presented regular black holes with multi-horizons in modified gravity with a nonlin-

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ear Maxwell field. The solution-generating method means that one presupposes the metric functions initially and then solves for the Maxwell field and the Lagrangian. Also using the solution-generating method, we obtained the solutions for static and spherically symmetric black holes and cosmological spacetimes with multiple horizons and multiple singularities in the generalized Einstein–Maxwell theory [9]. On the other hand, by using the method of solving the equations of motion straightforwardly, we find the multi-horizon black hole solutions based on the Lagrangian with a series of Maxwell terms, $F_{\mu\nu}F^{\mu\nu}$ [10]. In this paper, we shall extend the study to a more general case. Specifically, we take the Lagrangian for the Maxwell field as a series of $\sqrt{-F_{\mu\nu}F^{\mu\nu}/2}$, rather than $F_{\mu\nu}F^{\mu\nu}$. The former is obviously different from the latter. One of the benefits of the extension is the emergence of some new terms in the line element. Those new terms modify the structure of spacetime significantly. Other solutions for black holes, black branes, magnetic black holes and magnetic branes with nonlinear Maxwell field can be found in Refs. [11–37].

The paper is organized as follows. In Sect. 2, starting from the most general Lagrangian for the nonlinear Maxwell field, we derive multiple-horizon black holes by using the series-expansion method. In Sect. 3, using the solution-generating method, we obtain the Lagrangian for three-horizon black hole solution with and without the cosmological term. In Sect. 4, we explore the thermodynamics of 7-horizon black holes. In practice, the first law of thermodynamics and the Smarr relation are found. In Sect. 5 and Sect. 6, we consider the motion of test particles and test fields in the background of multiple-horizon spacetimes. In the last section, we present solutions without mass inflation. Throughout the paper, we adopt the units in which $G = c = \hbar = 1$ and the metric signature $(-, +, +, +)$.

2 Exact solutions

In this section, we seek exact solutions for multi-horizon black holes in Einstein nonlinear-Maxwell theory. The theory we consider has the following four-dimensional action:

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{4}R - \frac{1}{4}F^2 + \sum_{n=3}^{\infty} a_n \chi^n \right), \quad (2)$$

where

$$F^2 = F^{\mu\nu}F_{\mu\nu}, \quad \chi = \sqrt{-\frac{1}{2}F^2},$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad (3)$$

with A_μ the Maxwell four-vector field. R is the Ricci scalar. The action is gauge invariant under the transformation,

$$A_\mu \rightarrow A_\mu + \nabla_\mu \psi, \quad (4)$$

with ψ an arbitrary scalar field. The Lagrangian L for the nonlinear Maxwell field considered here is different from that in Ref. [10] where the Lagrangian is a series of Maxwell terms, $F_{\mu\nu}F^{\mu\nu}$. The corresponding Lagrangian in that case is equivalent to

$$L = -\frac{1}{4}F^2 + \sum_{n=3}^{\infty} a_n \chi^n + \lambda (2\chi^2 + F^2), \quad (5)$$

where λ plays the role of an auxiliary scalar field.

It has been shown [9] that a four-dimensional static spherically symmetric spacetime sourced by a nonlinear Maxwell field can always be written as

$$ds^2 = -U(r) dt^2 + U(r)^{-1} dr^2 + r^2 d\Omega_2^2. \quad (6)$$

Namely, there is only one freedom of $U(r)$ to be determined. Since the spacetime is static and spherically symmetric, the four-vector can be assumed as

$$[A_\mu] = [\phi(r), 0, 0, 0]. \quad (7)$$

Here ϕ corresponds to the Coulomb potential in Maxwell electrostatics. Let

$$\phi = \int K(r) dr, \quad (8)$$

we obtain the following independent equations from Einstein equations

$$r^2 K + \sum_{n=3}^{\infty} n r^2 a_n K^{n-1} + Q = 0, \quad (9)$$

$$\frac{1}{2} - \frac{1}{2}U - \frac{1}{2}rU' - \frac{1}{2}K^2 r^2 - \sum_{n=3}^{\infty} (n-1) r^2 a_n K^n = 0. \quad (10)$$

K corresponds to minus the electric field intensity in electrostatics. Q is an integration constant which has the physical meaning of the electric charge of the whole spacetime. We focus on the series solutions

$$K = \frac{c_2}{r^2} + \frac{c_3}{r^3} + \frac{c_4}{r^4} + \frac{c_5}{r^5} + \dots,$$

$$U = d_0 + \frac{d_1}{r} + \frac{d_2}{r^2} + \frac{d_3}{r^3} + \frac{d_4}{r^4} + \frac{d_5}{r^5} + \dots, \quad (11)$$

with c_i and d_i being constants to be determined. Substituting them into the equations of motion and requiring each term in the power series (from low to high) in r to vanish, We obtain

$$\begin{aligned}
 K &= -\frac{Q}{r^2} - \frac{3a_3Q^2}{r^4} - \frac{2Q^3(9a_3^2 - 2a_4)}{r^6} \\
 &\quad - \frac{5Q^4(27a_3^3 - 12a_4a_3 + a_5)}{r^8} \\
 &\quad - \frac{6Q^5(-a_6 + 15a_5a_3 + 189a_3^4 - 126a_4a_3^2 + 8a_4^2)}{r^{10}} \\
 &\quad - \frac{7Q^6}{r^{12}} \left(a_7 - 1296a_3^3a_4 + 192a_3a_4^2 - 20a_5a_4 \right. \\
 &\quad \left. - 18a_3a_6 + 180a_3^2a_5 + 1458a_3^5 \right) + \dots, \\
 U &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{2a_3Q^3}{3r^4} + \frac{Q^4(9a_3^2 - 2a_4)}{5r^6} \\
 &\quad + \frac{2Q^5(27a_3^3 - 12a_4a_3 + a_5)}{7r^8} \\
 &\quad + \frac{2Q^6(-a_6 + 15a_5a_3 + 189a_3^4 - 126a_4a_3^2 + 8a_4^2)}{9r^{10}} \\
 &\quad + \frac{2Q^7}{11r^{12}} \left(a_7 - 1296a_3^3a_4 + 192a_3a_4^2 - 20a_5a_4 \right. \\
 &\quad \left. - 18a_3a_6 + 180a_3^2a_5 + 1458a_3^5 \right) + \dots. \quad (12)
 \end{aligned}$$

Here M and Q are the physical mass and electric charge, respectively. Note that we have used the following relations, $c_2 = -Q$, $c_3 = 0$, $c_4 = -3a_3Q^2$, etc. Comparing the two expressions of K and U carefully, we find that the ratio of every two terms with the same power of r is proportional to the charge, Q . In other words, every such pair of terms has a common constant factor. More specifically, we observe that a_3 appears in the r^4 terms, a_4 (and a_3) appears in the r^6 terms, a_5 (and a_4, a_3) appears in the r^8 terms, a_6 (and a_5, a_4, a_3) appears in the r^{10} terms, and so on. This tells us we can remove high-order terms of r to any desired order by appropriately choosing the parameters. For example, if we let

$$\begin{aligned}
 a_4 &= \frac{9}{2}a_3^2, \quad \Leftarrow r^6 \text{ term}, \\
 a_5 &= -27a_3^3 + 12a_4a_3, \quad \Leftarrow r^8 \text{ term}, \\
 a_6 &= 15a_5a_3 + 189a_3^4 - 126a_4a_3^2 + 8a_4^2, \quad \Leftarrow r^{10} \text{ term}, \\
 \dots &= \dots, \quad (13)
 \end{aligned}$$

we would get

$$K = -\frac{Q}{r^2} - \frac{3a_3Q^2}{r^4}, \quad (14)$$

$$U = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{2a_3Q^3}{3r^4}. \quad (15)$$

We will see that this spacetime describes a three-horizon black hole. As another example, if we let

$$\begin{aligned}
 a_8 &= \mathcal{F}_8(a_7, a_6, a_5, a_4, a_3), \quad \Leftarrow r^{14} \text{ term}, \\
 a_9 &= \mathcal{F}_9(a_8, a_7, a_6, a_5, a_4, a_3), \quad \Leftarrow r^{16} \text{ term}, \\
 \dots &= \dots, \quad (16)
 \end{aligned}$$

we obtain

$$\begin{aligned}
 K &= -\frac{Q}{r^2} - \frac{3a_3Q^2}{r^4} - \frac{2Q^3(9a_3^2 - 2a_4)}{r^6} \\
 &\quad - \frac{5Q^4(27a_3^3 - 12a_4a_3 + a_5)}{r^8} \\
 &\quad - \frac{6Q^5(-a_6 + 15a_5a_3 + 189a_3^4 - 126a_4a_3^2 + 8a_4^2)}{r^{10}} \\
 &\quad - \frac{7Q^6}{r^{12}} \left(a_7 - 1296a_3^3a_4 + 192a_3a_4^2 - 20a_5a_4 \right. \\
 &\quad \left. - 18a_3a_6 + 180a_3^2a_5 + 1458a_3^5 \right), \\
 U &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{2a_3Q^3}{3r^4} + \frac{Q^4(9a_3^2 - 2a_4)}{5r^6} \\
 &\quad + \frac{2Q^5(27a_3^3 - 12a_4a_3 + a_5)}{7r^8} \\
 &\quad + \frac{2Q^6(-a_6 + 15a_5a_3 + 189a_3^4 - 126a_4a_3^2 + 8a_4^2)}{9r^{10}} \\
 &\quad + \frac{2Q^7}{11r^{12}} \left(a_7 - 1296a_3^3a_4 + 192a_3a_4^2 - 20a_5a_4 \right. \\
 &\quad \left. - 18a_3a_6 + 180a_3^2a_5 + 1458a_3^5 \right). \quad (17)
 \end{aligned}$$

Here $\mathcal{F}_i$ denotes a combination of coefficients a_j . We will see in a later section that it describes in general a 7-horizon black hole. In all, the metric function U in Eq. (12) is the most general solution and can describe the black hole spacetime with an arbitrary number of horizons.

Finally, if we let

$$\begin{aligned}
 a_3 &= 0, \quad \Leftarrow r^4 \text{ term}, \\
 a_4 &\neq 0, \quad \Leftarrow r^6 \text{ term}, \\
 a_5 &= 0, \quad \Leftarrow r^8 \text{ term}, \\
 a_6 &= 8a_4^2, \quad \Leftarrow r^{10} \text{ term}, \\
 \dots &= \dots, \quad (18)
 \end{aligned}$$

we would get

$$\begin{aligned}
 K &= -\frac{Q}{r^2} + \frac{4Q^3a_4}{r^6}, \\
 U &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{2Q^4a_4}{5r^6}. \quad (19)
 \end{aligned}$$

To our surprise, the metric is nothing but the static solution for magnetically charged black holes in Einstein-Euler-Heisenberg gravity [38]. Therefore, it is fascinating that different theories give the same black hole solutions. Similarly, we can get many other solutions.

3 Three-horizon black holes with and without a cosmological constant

In this section, we study three-horizon black holes with and without the Einstein cosmological constant. Starting from the metric, we derive the corresponding Lagrangian of the theory.

3.1 The Lagrangian for three-horizon black holes without a cosmological constant

In this subsection, we construct the Lagrangian of the nonlinear Maxwell field for the three-horizon black hole solution. To this end, we start from the most general action

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{4} R + L(\chi) \right], \quad (20)$$

where $L(\chi)$ stands for an arbitrary function of χ . The action gives the Einstein equations

$$G_{\mu\nu} = -\frac{2}{\chi} \frac{dL}{d\chi} F_{\mu\lambda} F_{\nu}^{\lambda} - 2g_{\mu\nu} L, \quad (21)$$

and the equation of motion for the Maxwell field

$$\nabla_{\mu} \left(\frac{dL}{d\chi} F^{\mu\nu} \right) = 0. \quad (22)$$

In the background of the static and spherically symmetric spacetime given by Eq. (6), they are reduced to

$$r^2 L_{,\chi} + Q = 0, \quad (23)$$

and

$$\frac{1}{2} - \frac{1}{2} U - \frac{1}{2} U' r - r^2 \chi^2 \left(\frac{L}{\chi} \right)_{,\chi} = 0, \quad (24)$$

where Q is an integration constant. Comparing the two equations with Eq. (9) and Eq. (10), we recognize that K is exactly χ in the background of spacetime given by Eq. (6)

$$\chi = K. \quad (25)$$

Now we have three functions, U , χ and L , but two independent equations. Thus the equations of motion are not closed. Given anyone one of them in advance, then we can solve for the other two. This is exactly the reason why we can reconstruct the Lagrangian for the three-horizon black holes. Substituting Eqs. (14) and (15) into above equations of motion, we obtain

$$L = \frac{18a_3\chi - 1 + \sqrt{-(12a_3\chi - 1)^3}}{108a_3^2}. \quad (26)$$

Using the L'Hospital's rule twice, we find that it reduces to $L = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu}$ which is the Maxwell case in the limit $a_3 = 0$. This is precisely the Lagrangian for three-horizon black holes without a cosmological constant. We notice that there is only one parameter a_3 in the theory. Both the mass M and charge Q are absent. This is a very nice character. We have checked that, starting from this Lagrangian, we would inevitably get the three-horizon solution by solving the equations of motion straightforwardly. When $a_3 < 0$, the metric function U gives, in general, three horizons, $\tilde{h}_3 > \tilde{h}_2 > \tilde{h}_1 > 0$. As an example, we have

$$\tilde{h}_1 = 0.08, \quad \tilde{h}_2 = 0.38, \quad \tilde{h}_3 = 1.60, \quad (27)$$

when

$$M = 1, \quad Q = 0.8, \quad a_3 = -0.1. \quad (28)$$

For some specific parameters, two or three horizons would coincide. To show this point, we factorize the function U as follows:

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{\frac{2}{3}a_3Q^3}{r^4} = \left(1 + \frac{\tilde{k}_1}{r}\right) \left(1 - \frac{\tilde{h}_1}{r}\right)^2 \left(1 - \frac{s\tilde{h}_1}{r}\right). \quad (29)$$

Then we arrive at

$$\begin{aligned} \tilde{k}_1 &= \frac{s\tilde{h}_1}{1+2s}, \quad a_3 = -\frac{3}{2}s^2\tilde{h}_1 \sqrt{\frac{1+2s}{(3s^2+2s+1)^3}}, \\ M &= \frac{\tilde{h}_1(1+s)^2}{1+2s}, \quad Q = \tilde{h}_1 \sqrt{\frac{3s^2+2s+1}{1+2s}}. \end{aligned} \quad (30)$$

Let us assume that

$$\tilde{h}_1 > 0, \quad s > 0. \quad (31)$$

Then we have the following conclusions.

(1). When

$$s = 1, \quad (32)$$

the three horizons $\tilde{h}_3$, $\tilde{h}_2$ and $\tilde{h}_1$ coincide.

(2). When

$$s > 1, \quad (33)$$

the two inner horizons $\tilde{h}_2$ and $\tilde{h}_1$ coincide.

(3). When

$$0 < s < 1, \quad (34)$$

the event horizon $\tilde{h}_3$ coincides with the inner horizon $\tilde{h}_2$.

3.2 Lagrangian for the three-horizon black holes with a cosmological constant

In this subsection, we present the Lagrangian for the three-horizon black holes with cosmological constant. After obtaining the Lagrangian Eq. (35) for three-horizon black holes without a cosmological constant, it is straightforward to extend it to the case with a cosmological constant by simply adding the term for Λ

$$L = \frac{18a_3\chi - 1 + \sqrt{-(12a_3\chi - 1)^3}}{108a_3^2} - \frac{1}{2}\Lambda. \quad (35)$$

Starting from the above Lagrangian, we obtain the black hole solution as follows

$$K = -\frac{Q}{r^2} - \frac{3a_3Q^2}{r^4},$$

$$U = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{2a_3Q^3}{3r^4} - \frac{1}{3}\Lambda r^2. \quad (36)$$

In this case, there are generally four horizons for positive cosmological constant in this spacetime, one cosmological horizon and three black hole horizons. It is straightforward to obtain the the solutions for Eqs. (12), (17) and (19) with a cosmological constant by simply adding the term $-\frac{1}{3}\Lambda r^2$ to the function U .

4 Thermodynamics of a 7-horizon black hole

In this section, we shall study the thermodynamics of multiple horizons. As an example, we consider the case of a 7-horizon. We find that positive and negative surface gravities appear alternately. It is exactly this feature that allows us to construct black holes without mass inflation.

The solution for a 7-horizon black holes can be written from Eq. (17) as

$$\phi = \frac{Q}{r} + \frac{a_3Q^2}{r^3} + \frac{2Q^3\tilde{a}_4}{5r^5} + \frac{5Q^4\tilde{a}_5}{7r^7} + \frac{2Q^5\tilde{a}_6}{3r^9} + \frac{7Q^6\tilde{a}_7}{11r^{11}}, \quad (37)$$

and

$$U = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{\frac{2}{3}a_3Q^3}{r^4} + \frac{\frac{1}{5}Q^4\tilde{a}_5}{r^6} + \frac{\frac{2}{7}Q^5\tilde{a}_6}{r^8} + \frac{\frac{2}{9}Q^6\tilde{a}_6}{r^{10}} + \frac{\frac{2}{11}Q^7\tilde{a}_7}{r^{12}}. \quad (38)$$

Here $\tilde{a}_i$ are the rescaled a_i . The spacetime can have at most seven horizons. For example, when

$$M = 1, \quad Q = 0.9, \quad a_3 = -0.01, \quad \tilde{a}_4 = 10^{-5},$$

$$\tilde{a}_5 = -10^{-10}, \quad \tilde{a}_6 = 10^{-16}, \quad \tilde{a}_7 = -10^{-23}, \quad (39)$$

we obtain from the horizon equation $U = 0$

$$r_i = -0.069, \quad -0.0164, \quad -0.00357, \quad -0.0008, \\ -0.00029, \quad 0.000289, \quad 0.000809, \quad 0.00357, \\ 0.0164, \quad 0.085, \quad 0.546, \quad 1.44. \quad (40)$$

The seven positive roots correspond to the positions of seven horizons. Thus, we factorize U as follows:

$$U = \prod_{\alpha=1}^5 \left(1 + \frac{k_\alpha}{r}\right) \prod_{\alpha=1}^7 \left(1 - \frac{h_\alpha}{r}\right), \quad (41)$$

with k_i and h_i positive constants and h_i satisfying

$$0 < h_1 < h_2 < h_3 < h_4 < h_5 < h_6 < h_7. \quad (42)$$

Cvetič et al. [39] showed that if one regards the Christodoulou and Ruffini formula for the total energy or enthalpy as defining the Gibbs surface, then the rules of Gibbsian thermodynamics imply that negative temperatures arise inevitably on some inner horizons, as does the conventional form of the first law. They propose the Gibbsian temperature of horizon- i is given by [39,40]

$$T_i = \frac{1}{4\pi} U'|_{r=h_i} = \frac{1}{4\pi h_i} \prod_{\alpha=1}^5 \left(1 + \frac{k_\alpha}{h_i}\right) \prod_{\alpha \neq i}^7 \left(1 - \frac{h_\alpha}{h_i}\right). \quad (43)$$

So we have

$$T_7 = 8.51 \cdot 10^{-15} \left(\frac{10^8 M_\odot}{M}\right) \text{K},$$

$$T_6 = -5.70 \cdot 10^{-14} \left(\frac{10^8 M_\odot}{M}\right) \text{K},$$

$$T_5 = 3.54 \cdot 10^{-11} \left(\frac{10^8 M_\odot}{M}\right) \text{K},$$

$$T_4 = -1.42 \cdot 10^{-7} \left(\frac{10^8 M_\odot}{M}\right) \text{K},$$

$$T_3 = 6.02 \cdot 10^{-3} \left(\frac{10^8 M_\odot}{M}\right) \text{K},$$

$$T_2 = -3.54 \cdot 10^3 \left(\frac{10^8 M_\odot}{M}\right) \text{K},$$

$$T_1 = 3.07 \cdot 10^8 \left(\frac{10^8 M_\odot}{M}\right) \text{K}. \quad (44)$$

We see that some of the inner horizons have positive temperature while some have negative temperature. It is well-known that absolute temperature is classically required to be positive. However, under some special quantum conditions, negative temperatures – in which high-energy states

are more occupied than low-energy states – are also possible. Such states have been realized in localized spin systems [41–43]. Thus the inner horizons with negative temperature may be thought of as analogous to a spin system. Volovik [44] argued that every positive temperature determines the tunneling rate from $r_* = -\infty$ to $r_* = +\infty$, while every negative temperature determines the occupation number of particles in the vicinity of $r_* = -\infty$. Physically, the sign of the temperature for the horizon is closely related to the fact that an attractive force or repulsive force detected by an observer from the horizon. The deeper we go toward the center of the black hole, the higher the magnitude of temperature for the inner horizons. For black holes with many horizons, it can be inferred that the temperature of the innermost horizon will be astonishingly high.

In theory, the original suggestion that inner horizons should be assigned a negative temperature [45] was based not on quantum field theory, but on quantum mechanical systems, such as spin systems, exhibiting population inversion [46, 47]. This proposal of negative temperature is consistent with the laws of thermodynamics [48, 49]. On the other hand, through the correlation between temperature and surface gravities, Tiandho [50] concluded that the temperature of the inner horizon is always negative and that of the outer horizon is always positive for Reissner–Nordstrom black holes. Wu [51] also proposed that the inner horizon's temperature is negative because of its negative surface gravity. Finally, experimentally, by tailoring the Bose–Hubbard Hamiltonian, S. Braun et al. [52] created an attractively interacting ensemble of ultra-cold bosons at negative temperature.

Now we turn to the Bekenstein–Hawking entropy which is associated with horizon- i as follows

$$S_i = \pi h_i^2. \quad (45)$$

The corresponding Coulomb potential on horizon- i is

$$\Phi_i = \phi|_{r=h_i}. \quad (46)$$

In order to derive the first law of thermodynamics, we focus on the horizon equation.

$$U|_{r=h_i} = 0, \quad (47)$$

from which we obtain

$$M = \frac{1}{2}h_i + \frac{Q^2}{2h_i} + \frac{\frac{1}{3}a_3Q^3}{h_i^3} + \frac{\frac{1}{10}Q^4\tilde{a}_4}{h_i^5} + \frac{\frac{1}{7}Q^5\tilde{a}_5}{h_i^7} + \frac{\frac{1}{9}Q^6\tilde{a}_6}{h_i^9} + \frac{\frac{1}{11}Q^7\tilde{a}_7}{h_i^{11}}, \quad (48)$$

or

$$M = \frac{1}{2}\left(\frac{S_i}{\pi}\right)^{\frac{1}{2}} + \frac{Q^2}{2}\left(\frac{\pi}{S_i}\right)^{\frac{1}{2}} + \frac{a_3Q^3}{3}\left(\frac{\pi}{S_i}\right)^{\frac{3}{2}}$$

$$+ \frac{Q^4\tilde{a}_4}{10}\left(\frac{\pi}{S_i}\right)^{\frac{5}{2}} + \frac{Q^5\tilde{a}_5}{7}\left(\frac{\pi}{S_i}\right)^{\frac{7}{2}} + \frac{Q^6\tilde{a}_6}{9}\left(\frac{\pi}{S_i}\right)^{\frac{9}{2}} + \frac{Q^7\tilde{a}_7}{11}\left(\frac{\pi}{S_i}\right)^{\frac{11}{2}}. \quad (49)$$

Now the energy M is regarded as a function of the extensive variables, S_i , Q , a_3 , $\tilde{a}_4$, $\tilde{a}_5$, $\tilde{a}_6$, $\tilde{a}_7$. Differentiating the above relation, we find the first-law of thermodynamics

$$\begin{aligned} dM &= \frac{\partial M}{\partial S_i}dS_i + \frac{\partial M}{\partial Q}dQ + \frac{\partial M}{\partial a_3}da_3 + \frac{\partial M}{\partial \tilde{a}_4}d\tilde{a}_4 \\ &\quad + \frac{\partial M}{\partial \tilde{a}_5}d\tilde{a}_5 + \frac{\partial M}{\partial \tilde{a}_6}d\tilde{a}_6 + \frac{\partial M}{\partial \tilde{a}_7}d\tilde{a}_7 \\ &= T_idS_i + \Phi_idQ + J_3da_3 + J_4d\tilde{a}_4 + J_5d\tilde{a}_5 \\ &\quad + J_6d\tilde{a}_6 + J_7d\tilde{a}_7, \end{aligned} \quad (50)$$

where

$$\begin{aligned} J_3 &= \frac{Q^3}{3h_i^3}, \quad J_4 = \frac{Q^4}{10h_i^5}, \quad J_5 = \frac{Q^5}{7h_i^7}, \\ J_6 &= \frac{Q^6}{9h_i^9}, \quad J_7 = \frac{Q^7}{11h_i^{11}}, \end{aligned} \quad (51)$$

and T_i , Φ_i are exactly that defined previously. It is found that the Smarr relation

$$M = 2T_iS_i + \Phi_iQ + J_3a_3 + 2J_4\tilde{a}_4 + 3J_5\tilde{a}_5 + 4J_6\tilde{a}_6 + 5J_7\tilde{a}_7, \quad (52)$$

holds on every single horizon.

When the cosmological constant is taken into account, the first law is generalized to

$$dM = T_idS_i + \Phi_idQ + J_3da_3 + J_4d\tilde{a}_4 + J_5d\tilde{a}_5 + J_6d\tilde{a}_6 + J_7d\tilde{a}_7 + \mathbb{V}d\mathbb{P}, \quad (53)$$

where the pressure and the thermodynamic volume are defined as usual [53, 54]

$$\mathbb{P} = -\frac{\Lambda}{8\pi}, \quad \mathbb{V} = \frac{4\pi h_i^2}{3}, \quad (54)$$

The Smarr relation turns out to be

$$M = 2T_iS_i + \Phi_iQ + J_3a_3 + 2J_4\tilde{a}_4 + 3J_5\tilde{a}_5 + 4J_6\tilde{a}_6 + 5J_7\tilde{a}_7 - 2\mathbb{V}\mathbb{P}. \quad (55)$$

5 Motion of test particles

In this section, we consider the motion of radially moving massive particles in the background of a multi-horizon black hole spacetime. To simplify our discussion, we will assume that the particles are electrically neutral and have unit rest mass. In this case, the particles follow geodesics. For electrically charged particles, one must take into account the Lorentz force on the particles produced by the black hole.

5.1 Classical motion

In this subsection, we consider the classical motion of test particles. The equation of motion for a neutral particle is then given by

$$\dot{r}^2 + U \left(1 + \frac{L^2}{r^2} \right) = E^2, \quad (56)$$

where E and L are the energy and angular momentum of the particle, respectively. This clearly shows that the sum of kinetic energy and potential energy is a constant where

$$V_{eff} = U \left(1 + \frac{L^2}{r^2} \right), \quad (57)$$

plays the role of an effective potential. A qualitative analysis of the radial motion of particles can be obtained by plotting the effective potential U , as shown in Fig. 1. Qualitative analysis indicates that the motion with angular momentum is similar to the radial motion. Therefore, it is sufficient to give only an analysis on radial motion. As an example, we consider 7-horizon and 8-horizon black holes, respectively, i.e. $h_1 < h_2 < h_3 < h_4 < h_5 < h_6 < h_7 < h_8$. For 7-horizon black holes, the potential approaches positive infinity in the vicinity of the singularity as shown by dotted line a . In contrast, the potential approaches negative infinity near the singularity as shown by dotted line b . We see there exist radial boundaries for the motion of the particle depending on the value of energy squared E^2 , as indicated with pointed lines. We shall make a discussion in three situations, i.e. $E^2 = 1$, $E^2 > 1$ and $E^2 < 1$, respectively. In the first place, the case $E^2 = 1$ corresponds to the particle being released from rest at infinity. For a 7-horizon black hole, the particle with energy $E^2 = 1$ crosses h_8, h_7, h_6, h_5, h_4 successively. Then it encounters a potential barrier and rebounds. Subsequently, the particle crosses the horizons h_4, h_5, h_6, h_7, h_8 in chronological order. In the end, it escapes to infinity again. We emphasize that this process is only experienced by the free-falling observer. The distant observer far away from the black hole will never observe this process. On the other hand, if the particle is inside the black hole, it would oscillate in some region. This physical process is also only undergone by the particle itself and the distant observer is unable to detect the process. Secondly, when $E^2 > 1$ and for 7-horizon black hole, the particle always cannot reach the central singularity but is instead repelled, passing back through all the horizons and ultimately emerges in another asymptotically flat spacetime which is different from ours. For sufficiently large energy $E^2 > 1$ and for 8-horizon black hole, the particle eventually falls into the singularity. Finally, we consider the case of $E^2 < 1$. In this case, for both 7-horizon and 8-horizon black holes, there always exists local minimum and local

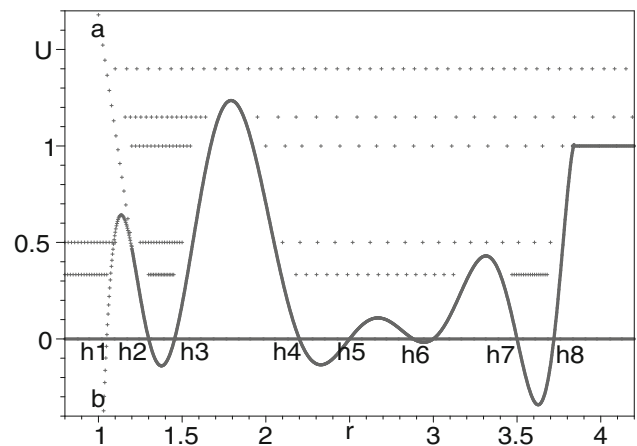


Fig. 1 Schematic behavior of the effective potential U for zero angular momentum. The two dotted branches labeled a and b illustrate the two possible near-singularity behaviors discussed in the text

maximum of the radial motion for particles. This indicates that those neutral particles always oscillate between their maximum and minimum distances. In this situation, many particles are bounded in the interior of black holes which is very similar to that electrons being bounded in an atom. We emphasize once again that due to the presence of event horizon, all the precesses occurring in the interior of the black hole will never be seen by a distant observer. However, it is not the case from the viewpoint of quantum mechanics.

5.2 Quantum motion

In this subsection, we consider the quantum motion of test particles. Eq. (56) describes a one-dimensional classical motion of test particles. By contrast, the equation of motion for the quantum motion is given by the stationary Schrodinger equation

$$-\frac{1}{2} \frac{d^2 \Psi}{dr^2} + V_{eff} \Psi = \omega \Psi, \quad (58)$$

where ω is the energy of the particle. We note that here the energy of a quantum particle can be negative, i.e. $\omega < 0$. However, the energy of a classical particle discussed in the above subsection is always positive, i.e. $E > 0$. Because there are many potential wells, the bound states for the particle exist and the discrete energy levels of particles can be given by the WKB method,

$$\left(n + \frac{1}{2} \right) \pi = \int_a^b \sqrt{2(\omega - V_{eff})} dr. \quad (59)$$

Here a and b stand for the turning points. On the other hand, there are also many potential barriers. Using the WKB method, the transmission probability of a particle is given by

$$\mathbb{T} = e^{-2 \left| \int_a^b \sqrt{2(\omega - V_{eff})} dr \right|}. \quad (60)$$

Therefore, from the viewpoint of quantum mechanics, a particle inside the black hole can escape from the black hole to spatial infinity.

6 The motion of the Klein–Gordon field

In this section, we consider the motion of Klein–Gordon field in the background of a multi-horizon black hole spacetime. The field is regarded as a test scalar field. This means that the field makes no contribution to the curvature of the spacetime manifold. By studying the motion of test scalar field, we can obtain information about the curved spacetime background just like we can solve for the solar mass from the motion of the Earth. The equation of motion for the Klein–Gordon field is

$$\nabla_\mu \nabla^\mu \varphi - \mu_0^2 \varphi = 0. \quad (61)$$

where μ_0 is the mass of test scalar field. Separating φ as follows:

$$\varphi = \frac{e^{-i\omega t} \psi(r)}{r} Y_{lm}, \quad (62)$$

with Y_{lm} the spherical harmonic function. Defining the tortoise coordinate

$$\begin{aligned} r_* &= \int \frac{dr}{U} \\ &= r - \sum_{\beta=1}^5 \frac{k_\beta^{12} \ln |r + k_\beta|}{\prod_{\alpha=1(\alpha \neq \beta)}^5 (k_\beta - k_\alpha) \prod_{\alpha=1}^7 (k_\beta + h_\alpha)} \\ &\quad + \sum_{\beta=1}^7 \frac{h_\beta^{12} \ln |r - h_\beta|}{\prod_{\alpha=1}^5 (h_\beta + k_\alpha) \prod_{\alpha=1(\alpha \neq \beta)}^7 (h_\beta - h_\alpha)}. \end{aligned} \quad (63)$$

We conclude that

$$\begin{aligned} r = +\infty &\Leftrightarrow r_* = +\infty, \\ r = h_7 &\Leftrightarrow r_* = -\infty, \\ r = h_6 &\Leftrightarrow r_* = +\infty, \\ r = h_5 &\Leftrightarrow r_* = -\infty, \\ r = h_4 &\Leftrightarrow r_* = +\infty, \\ r = h_3 &\Leftrightarrow r_* = -\infty, \\ r = h_2 &\Leftrightarrow r_* = +\infty, \\ r = h_1 &\Leftrightarrow r_* = -\infty, \\ r = 0 &\Leftrightarrow r_* = 0. \end{aligned} \quad (64)$$

Therefore, the space is divided into eight regions, i.e.

$$\begin{aligned} (1) & (+\infty, h_7], & (2) & [h_7, h_6], \\ (3) & [h_6, h_5], & (4) & [h_5, h_4], \\ (5) & [h_4, h_3], & (6) & [h_3, h_2], \\ (7) & [h_2, h_1], & (8) & [h_1, 0). \end{aligned} \quad (65)$$

In the tortoise coordinate system, the equation of motion takes the form

$$\frac{\partial^2 \psi}{\partial r_*^2} = (V - \omega^2) \psi, \quad (66)$$

where the prime denotes the derivative with respect to r . The potential V is defined as

$$V = U \left[\frac{U'}{r} + \frac{l(l+1)}{r^2} + \mu_0^2 \right]. \quad (67)$$

This equation is similar in form to the stationary Schrodinger equation after we regard r as a function of r_* . Without loss of generality, we consider the massless case $\mu_0 = 0$. Then the potential vanishes in both spatial infinity and all the horizons. Thus we can solve the equation of motion in every region with the boundary conditions

$$\psi \sim e^{\pm i\omega r_*}, \quad (68)$$

at spatial infinity and at all the horizons. Here $+$ and $-$ denotes the purely outgoing and ingoing waves, respectively. From Figs. 2, 3, 4, 5, 6, 7, 8 and 9, we plot the potential for every region.

As shown in Fig. 2, the potential in region (1) or $[h_7, +\infty)$, is always positive and behaves as a barrier which becomes higher as the angular quantum number l increases. There are four important processes in this region. In the first place, quantum perturbations in this region are stable because the potential is positive everywhere. This was shown by Vincent Moncrief in the case of Reissner–Nordstrom black hole [55]. Second, reflection and transmission phenomena will occur in this region. Third, Hawking radiation is produced in this region. Finally, the quasinormal modes are also produced here.

As shown in Fig. 3, the potential in region (2) or $[h_6, h_7]$ behaves as a well which gets even deeper with increasing angular quantum number l . In this region, three important processes occur. The first one is the transition and reflection and the second is the existence of bound states of waves. The third is the trapping of fields. The trapping process can be outlined as follows. The equation of motion Eq. (66) tells us that if the energy ω is purely imaginary

$$\omega = i\omega_0, \quad (69)$$

with ω_0 a positive constant and

$$V - \omega^2 = V + \omega_0^2 > 0, \quad (70)$$

the fields decay exponentially in r_* as $|r_*| = +\infty$ with the boundary conditions

$$\begin{aligned} \psi &\sim e^{-i\omega r_*} = e^{\omega_0 r_*} \sim 0, & \text{at } r_* = -\infty, \\ \psi &\sim e^{+i\omega r_*} = e^{-\omega_0 r_*} \sim 0, & \text{at } r_* = +\infty. \end{aligned} \quad (71)$$

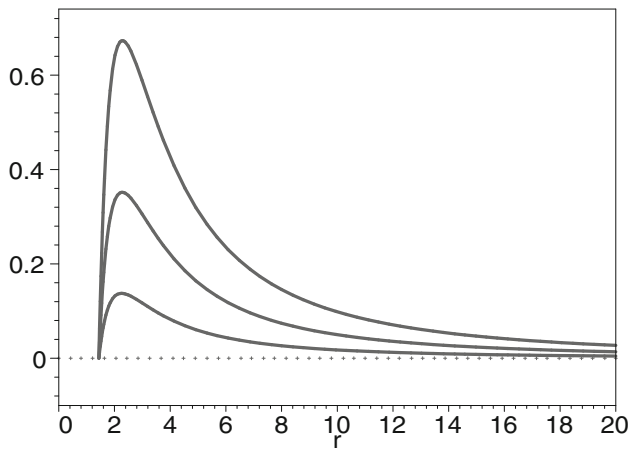


Fig. 2 There are potential barriers of order 10^0 in $[h_7, +\infty)$. The curves for $l = 3, 2, 1$ are shown from top to bottom, respectively

This means that the fields are trapped in the potential well. On the other hand, the fields decrease exponentially (the growing mode is unphysical and is therefore discarded) in time which means that they are absorbed by the potential well. With the trapping of fields in the well, this region becomes increasingly unstable. Eventually, the potential well is filled and leveled up by the accumulated energy. Then the inner horizon disappears.

As shown in Fig. 4, the potential in region (3) behaves as a barrier for large l . For small l , a potential well gradually appears and coexists with the barrier. The emergence of negative gap in the potential leads to eikonal instability according to [56]. We expect the sinking and inflation of waves would happen in the well. We note that the height of the potential is 5 orders of magnitude of region (1) and region (2). Similar cases occur for the potentials in regions (4)–(7) as shown in Figs. 5, 6, 7 and 8. As shown in Fig. 9, the potential in Region (8) gives us an infinite deep well. It is remarkable that the effect of l is negligible in regions (6, 7, 8).

Now let us consider a wave incident on the black hole from spatial infinity. First, the wave encounters a potential barrier with a height order of 10^0 near the event horizon. It undergoes reflection and transmission. The wave that passes through h_7 then encounters a potential well with a depth order of 10^0 and undergoes reflection and transmission again. In this region, the capture of waves would happen. The surviving wave then passes through h_6 and encounters a high potential barrier (on the order of 10^5), continuing to undergo reflection and transmission. For small angular quantum numbers, the wave first encounters a potential well, followed by a potential barrier. After passing through the horizon h_5 , the wave will encounter a deeper potential well, and after passing through the horizon h_4 , the wave will encounter a higher potential barrier again. The further inside, the less significant the effect of the angular quantum numbers becomes. In fact,

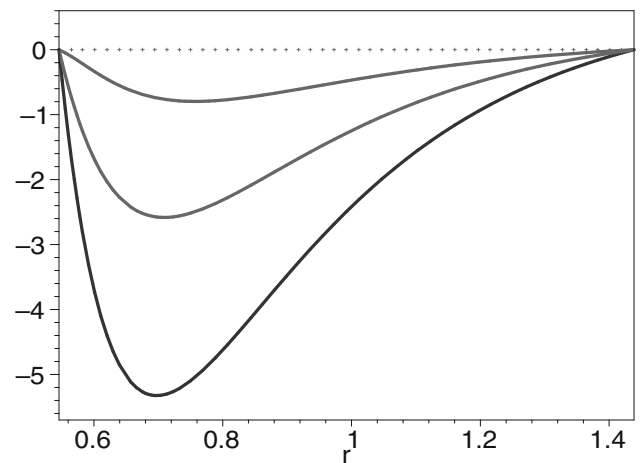


Fig. 3 There are potential wells of order 10^0 in $[h_6, h_7]$. The curves for $l = 1, 2, 3$ are shown from top to bottom, respectively

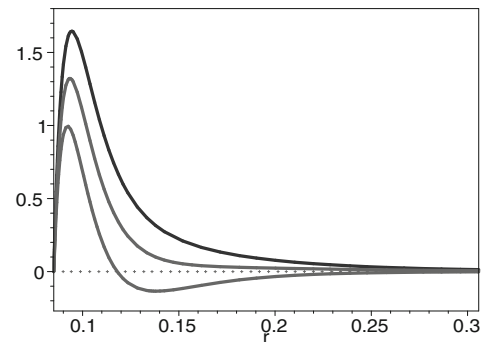


Fig. 4 Potential barriers coexist with potential wells of order 10^5 in $[h_5, h_6]$. The curves for $l = 7, 5, 1$ are shown from top to bottom, respectively

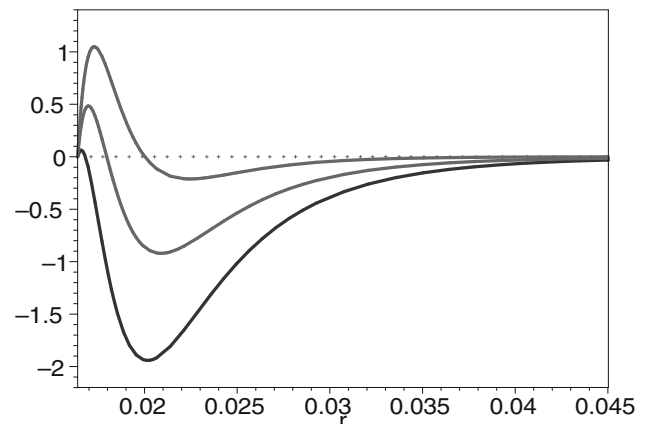


Fig. 5 Potential barriers coexist with potential wells of order 10^{12} in $[h_4, h_5]$. The curves for $l = 1, 200, 300$ are shown from top to bottom, respectively

the influence of large or small angular quantum numbers on the potential is no longer apparent. Finally, we see that the potential barrier becomes extremely high and the potential

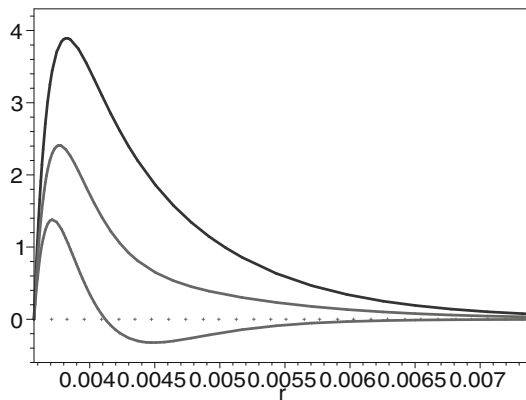


Fig. 6 Potential barriers coexist with potential wells of order 10^{21} in $[h_3, h_4]$. The curves for $l = 30,000, 20,000, 1$ are shown from top to bottom, respectively

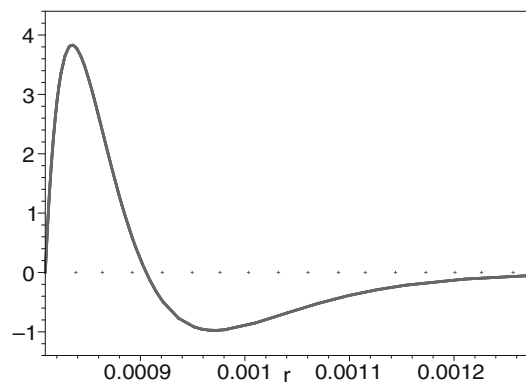


Fig. 7 Potential barriers coexist with potential wells of order 10^{32} in $[h_2, h_3]$. The cases $l = 1, 1000, 10,000$ are shown. This indicates that the effect of l on the potential is negligible

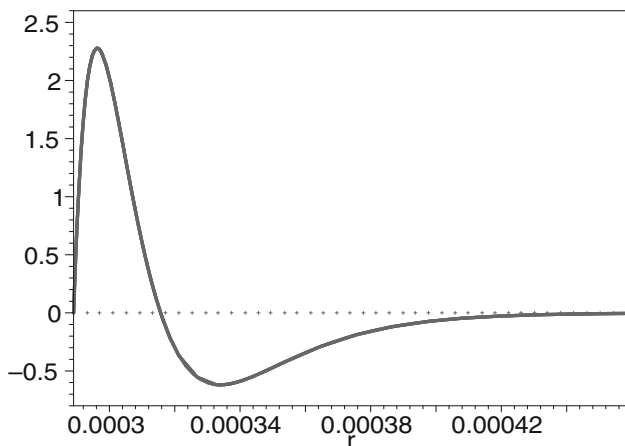


Fig. 8 Potential barriers coexist with potential wells of order 10^{42} in $[h_1, h_2]$. The cases $l = 1, 1000, 10,000$ are shown. This indicates that the effect of l on the potential is negligible

well becomes extremely deep. As the subsequent potential barriers become higher and higher, it will become increas-

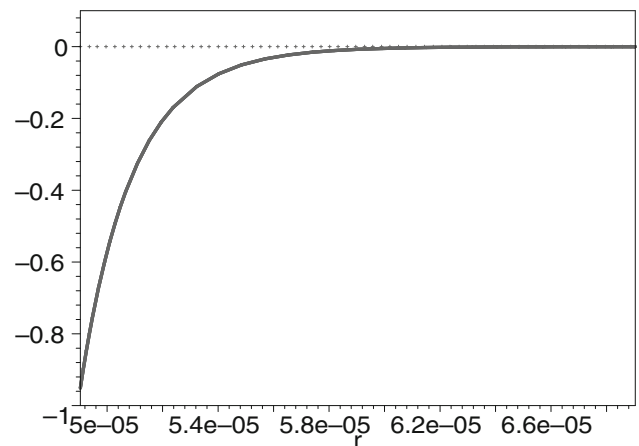


Fig. 9 Infinitely deep potential wells of order 10^{65} in $[0, h_1]$. The cases $l = 1, 1000, 10,000$ are shown. This indicates that the effect of l on the potential is negligible

ingly difficult for the wave to reach the center of the black hole.

7 Avoiding the mass inflation problem

In the above sections, we have derived and studied the physics of multi-horizon black holes. We find that many interesting phenomena occur in multi-horizon black holes, such as the alternate occurrence of positive and negative temperatures, the alternate occurrence of large potential barriers and potential wells, and the various interesting motions for classical particles. Although these physical features are interesting, but the solutions suffer from the mass inflation problem. So in the last section, we shall improve the multi-horizon black-hole solutions so that they avoid mass inflation by adjusting the inner horizons to overlap. We summarize the results below.

7.1 Three-horizon black holes as the mother solution

1. One-horizon black holes

The metric function of one-horizon black holes is given by

$$U = \left(1 + \frac{M}{4r}\right) \left(1 - \frac{3M}{4r}\right)^3, \quad (72)$$

with M the mass of the black hole. The solution comes from the three-horizon black hole solution with the three horizons coinciding, Eq. (32). In this case, both the surface gravity and temperature of the single horizon vanish. The singularity is spacelike.

2. Two-horizon black holes

The metric function of two-horizon black holes is given by

$$U = \left[1 + \frac{s\tilde{h}_1}{(1+2s)r} \right] \left(1 - \frac{\tilde{h}_1}{r} \right)^2 \left(1 - \frac{s\tilde{h}_1}{r} \right), \quad (73)$$

with $\tilde{h}_1 > 0$ and $s > 1$. It also arises from the three-horizon black holes, Eq. (33). In this case, we have two horizons. One of them is the outer event horizon and the other is the inner coincident horizon. Both the surface gravity and temperature of the inner horizon are zeros. While the outer event horizon has positive surface gravity and temperature. The singularity is spacelike.

7.2 Four-horizon black holes as the mother solution

1. Two-horizon black holes

Now we study the metric function of four-horizon black holes

$$U = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{\bar{a}_3}{r^4} + \frac{\bar{a}_4}{r^6}. \quad (74)$$

It has four parameters, M , Q , $\bar{a}_3$, $\bar{a}_4$. We can factorize it in the form

$$U = \left(1 + \frac{\bar{k}_1}{r} + \frac{\bar{k}_2}{r^2} \right) \left(1 - \frac{\bar{h}}{r} \right)^3 \left(1 - \frac{s\bar{h}}{r} \right). \quad (75)$$

Then we conclude that

$$\begin{aligned} \bar{k}_1 &= \frac{(1+3s)^2 \bar{h}}{3+8s^2+9s}, & \bar{a}_3 &= -\frac{\bar{h}^4 (3s+10s^3+6s^2+1)}{3+8s^2+9s}, \\ M &= \frac{4\bar{h}_1 (1+s)^3}{3+8s^2+9s}, & Q^2 &= \frac{3\bar{h}^2 (2+6s+7s^2+5s^3)}{3+8s^2+9s}, \\ \bar{k}_2 &= \frac{(1+3s)\bar{h}^2 s}{3+8s^2+9s}, & \bar{a}_4 &= \frac{s^2 \bar{h}^6 (1+3s)}{3+8s^2+9s}. \end{aligned} \quad (76)$$

Now the four-parameter black hole solution (M , Q , $\bar{a}_3$, $\bar{a}_4$) becomes a two-parameter one (s , $\bar{h}$). When

$$\bar{h} > 0, \quad s > 1, \quad (77)$$

the inner three horizons coincide with vanishing surface gravities. In this case, we are left with two horizons and the singularity is timelike. On the other hand, if

$$\bar{h} > 0, \quad 0 < s < 1, \quad (78)$$

the outer three horizons coincide with the vanishing surface gravities. In this case, the surface gravities (and the temperature) are positive for the surviving inner horizon. Therefore, the mass inflation phenomenon remains.

2. One-horizon black holes

When

$$\bar{h} > 0, \quad s = 1, \quad (79)$$

the inner three horizons and the outer event horizon coincide with the vanishing of surface gravities. In this case, the singularity is timelike.

3. Two-horizon black holes

Finally, we factorize the metric function as follows:

$$U = \left(1 + \frac{\bar{k}_1}{r} + \frac{\bar{k}_2}{r^2} \right) \left(1 - \frac{\bar{h}_1}{r} \right)^2 \left(1 - \frac{\bar{h}_2}{r} \right)^2, \quad (80)$$

and conclude that

$$\begin{aligned} \bar{k}_1 &= \frac{2(\bar{h}_1 + \bar{h}_2)\bar{h}_2\bar{h}_1}{\bar{h}_2^2 + 3\bar{h}_2\bar{h}_1 + \bar{h}_1^2}, & \bar{k}_2 &= \frac{\bar{h}_2^2\bar{h}_1^2}{\bar{h}_2^2 + 3\bar{h}_2\bar{h}_1 + \bar{h}_1^2}, \\ M &= \frac{(\bar{h}_1 + \bar{h}_2)^3}{\bar{h}_2^2 + 3\bar{h}_2\bar{h}_1 + \bar{h}_1^2}, & \bar{a}_4 &= \frac{\bar{h}_2^4\bar{h}_1^4}{\bar{h}_2^2 + 3\bar{h}_2\bar{h}_1 + \bar{h}_1^2}, \\ Q^2 &= \frac{\bar{h}_2^4 + \bar{h}_1^4 + 3\bar{h}_2^3\bar{h}_1 + 7\bar{h}_1^2\bar{h}_2^2 + 3\bar{h}_1^3\bar{h}_2}{\bar{h}_2^2 + 3\bar{h}_2\bar{h}_1 + \bar{h}_1^2}, \\ \bar{a}_3 &= -\frac{\bar{h}_1^2\bar{h}_2^2(\bar{h}_1\bar{h}_2 + 2\bar{h}_2^2 + 2\bar{h}_1^2)}{\bar{h}_2^2 + 3\bar{h}_2\bar{h}_1 + \bar{h}_1^2}, \end{aligned} \quad (81)$$

with two parameters

$$\bar{h}_1 > 0, \quad \bar{h}_2 > 0. \quad (82)$$

In this situation, we are left with two horizons both with the vanishing surface gravities. The singularity here is timelike.

Similarly, by taking the five-horizon black holes, six-horizon black holes, and so on as the mother solutions, one can produce many other black holes without mass inflation.

8 Conclusion and discussion

The well-known Reissner–Nordström black holes, Kerr black holes and Kerr–Newman black holes all suffer from the mass inflation problem. The superficial cause of the problem is that they have two horizons. The underlying cause is that an observer at rest on the event horizon feels an infinite attractive force, while an observer at rest on the inner horizon feels an infinite repulsive force. This is also the reason why the surface gravities of outer event horizon is positive while the surface gravities is negative for inner horizon. For free falling particles, the inner horizon is an insurmountable high wall. Therefore, particles falling into a black hole and the resulting energy density accumulate in the vicinity of the inner horizon. Thus the phenomenon of mass-inflation occurs. Noting that the mass inflation exponentially grows with the power of absolute value of surface gravities, Carballo-Rubio et al.

[4] showed that, if one is able to make the surface gravities of inner horizons vanish, then the exponential growth of mass inflation instability would not occur. It is indeed the case. In fact, the extremal black holes with two horizons coincident have the vanishing surface gravities such that the mass inflation does not exist. On the other hand, Cai [57] investigated the stability of the coincident Cauchy horizon. It was found that, despite the asymptotic behavior of the spacetime (flat, anti-de Sitter, or de Sitter), this kind of Cauchy horizon is stable against classical perturbations. In view of this, we began our exploration of multiple-horizon black holes.

In the framework of Einstein nonlinear-Maxwell field theories, we obtain black hole solutions in the most general form, (Eq. (12)). In the solution, two integration constants are present and they play the role of physical mass and electric charge, respectively. The other parameters a_i come from the theory. By adjusting the theoretical parameters a_i , the infinite series in the metric function becomes a finite one and various multiple-horizon black holes are found.

In the process of studying the thermodynamics of multiple horizon black holes, we find that the positive and negative surface gravities or temperatures, appear alternately. It is exactly this feature that enables us to construct black holes without mass inflation. According to the proposal of Cvetic et al. [39], these negative temperatures should be interpreted as Gibbsian temperatures.

By studying the motion of test particles and test fields, we find a phenomenon of alternating occurrence of a large number of potential barriers and potential wells. Therefore, under different boundary conditions, many quantum-mechanical phenomena, such as reflection, transmission, quantum bound states, and so on are expected to occur. In addition, various interesting motions of classical particles appear. In short, In short, the physics of multiple-horizon black holes is quite rich. Finally, as another strategy to avoid mass-inflation, Hale et al. [58] eliminate the Cauchy horizon by regularizing the self energy of point charges and Casadio et al. [59,60] used the method of horizon quantum mechanics to demonstrate that the probability of forming an inner horizon is lower than the probability of forming an outer event horizon.

To summarize, we construct the metrics for multiple-horizon black holes with the most general nonlinear Maxwell field by using the series-expansion method. Then the thermodynamics, the motion of classical and quantum fields, and the problem of mass inflation are explored for these black holes. However, the present study still has some limitations. In particular, observational signatures such as gravitational waves, shadows, light rings, and memory effects, as well as rotating extensions of the present solutions, remain interesting directions for future research.

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Code Availability Statement This manuscript has no associated code/software. [Author's comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

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References

1. E. Poisson, W. Israel, Internal structure of black holes. *Phys. Rev. D* **41**, 1796–1809 (1990)
2. A. Ori, Inner structure of a charged black hole: an exact mass-inflation solution. *Phys. Rev. Lett.* **67**, 789–792 (1991)
3. P.R. Brady, J.D. Smith, Black hole singularities: a numerical approach. *Phys. Rev. Lett.* **75**, 1256–1259 (1995)
4. R. Carballo-Rubio, F. Di Filippo, S. Liberati et al., Regular black holes without mass inflation instability. *JHEP* **2022**, 118 (2022)
5. A. Eichhorn, P.G.S. Fernandes, (2025). *arXiv preprint arXiv:2508.00686*
6. J. Lin, X. Zhang, M. Bravo-Gaete, *Phys. Rev. D* **111**(10), 106025 (2025)
7. M. Visser, Black holes, Cauchy horizons, and mass inflation. *Gen. Relativ. Gravit.* **56**, 146 (2024)
8. S. Nojiri, S.D. Odintsov, *Phys. Rev. D* **96**, 104008 (2017)
9. C. Gao, Y. Lu, S. Yu, Y. Shen, Black hole and cosmos with multiple horizons and multiple singularities in vector–tensor theories. *Phys. Rev. D* **97**, 104013 (2018)
10. C. Gao, Black holes with many horizons in the theories of nonlinear electrodynamics. *Phys. Rev. D* **104**, 064038 (2021)
11. E. Ayon-Beato, A. Garcia, *Phys. Rev. Lett.* **80**, 5056 (1998)
12. E. Ayon-Beato, A. Garcia, *Phys. Lett. B* **464**, 25 (1999)
13. E. Ayon-Beato, A. Garcia, *Gen. Relativ. Gravit.* **31**, 629 (1999)
14. E. Ayon-Beato, A. Garcia, *Gen. Relativ. Gravit.* **37**, 635 (2005)
15. N. Breton, *Phys. Rev. D* **67**, 124004 (2003)
16. R.G. Cai, D.W. Pang, A. Wang, *Phys. Rev. D* **70**, 124034 (2004)
17. R.G. Cai, Y.W. Sun, *JHEP* **09**, 115 (2008)
18. S.S. Yazadjiev, *Phys. Rev. D* **72**, 044006 (2005)
19. D.N. Vollick, *Phys. Rev. D* **72**, 084026 (2005)
20. Y.S. Myung, Y.W. Kim, Y.J. Park, *Phys. Rev. D* **78**, 044020 (2008)
21. Y.S. Myung, Y.W. Kim, Y.J. Park, *Phys. Rev. D* **78**, 084002 (2008)
22. S.H. Mazharimousavi, M. Halilsoy, Z. Amirabi, *Phys. Rev. D* **78**, 064050 (2008)
23. A. Khodam-Mohammadi, *Gravit. Cosmol.* **15**, 154 (2009)

24. M. Hassaine, C. Martinez, *Phys. Rev. D* **75**, 027502 (2007)
25. M. Hassaine, C. Martinez, *Class. Quantum Gravity* **25**, 195023 (2008)
26. H. Maeda, M. Hassaine, C. Martinez, *Phys. Rev. D* **79**, 044012 (2009)
27. S.H. Mazharimousavi, M. Halilsoy, *Phys. Lett. B* **681**, 190 (2009)
28. M.H. Dehghani, H.R. Sedehi, *Phys. Rev. D* **74**, 124018 (2006)
29. M.H. Dehghani, S.H. Hendi, A. Sheykhi, H.R. Sedehi, *JCAP* **02**, 020 (2007)
30. S.H. Hendi, *J. Math. Phys.* **49**, 082501 (2008)
31. K.A. Bronnikov, *Phys. Rev. D* **63**, 044005 (2001)
32. M.H. Dehghani, N. Bostani, S.H. Hendi, *Phys. Rev. D* **78**, 064031 (2008)
33. M.H. Dehghani, A. Sheykhi, S.H. Hendi, *Phys. Lett. B* **659**, 476 (2008)
34. K.A. Bronnikov, I.G. Dymnikova, E. Galaktionov, *Class. Quantum Gravity* **29**, 095025 (2012)
35. S.V. Bolokhov, K.A. Bronnikov, M.V. Skvortsova, *Class. Quantum Gravity* **29**, 245006 (2012)
36. K.A. Bronnikov, K.A. Baleevskikh, M.V. Skvortsova, *Phys. Rev. D* **96**, 124039 (2017)
37. J. Lin, M. Bravo-Gaete, X. Zhang, *Phys. Rev. D* **109**(10), 104039 (2024)
38. A. Allahyari, M. Khodadi, S. Vagnozzi, D.F. Mota, *JCAP* **02**, 003 (2020)
39. M. Cvetič, G.W. Gibbons, H. Lu, C.N. Pope, Killing horizons: negative temperatures and entropy super-additivity. *Phys. Rev. D* **98**, 106015 (2018)
40. Z. Zhai, W. Liu, Nernst theorem and Hawking radiation from a Reissner–Nordstrom black hole. *Astrophys. Space Sci.* **325**, 63–67 (2010)
41. E.M. Purcell, R.V. Pound, *Phys. Rev.* **81**, 279 (1951)
42. A.S. Oja, O.V. Lounasmaa, *Rev. Mod. Phys.* **69**, 1 (1997)
43. P. Medley, D.M. Weld, H. Miyake, D.E. Pritchard, W. Ketterle, *Phys. Rev. Lett.* **106**, 195301 (2011)
44. G.E. Volovik, Effect of the inner horizon on the black hole thermodynamics: Reissner–Nordström black hole and Kerr black hole. *Mod. Phys. Lett. A* **36**(24), 2150177 (2021)
45. A. Curir, Spin entropy of a rotating black hole. *II Nuovo Cimento B* **52**, 262–266 (1979)
46. N.F. Ramsey, Thermodynamics and statistical mechanics at negative absolute temperatures. *Phys. Rep.* **103**, 20 (1956)
47. E.M. Purcell, R.V. Pound, A nuclear spin system at negative temperature. *Phys. Rev.* **81**, 279 (1951)
48. R.H. Swendsen, J.S. Wang, Negative temperatures and the definition of entropy (2014). arXiv preprint [arXiv:1410.4619](https://arxiv.org/abs/1410.4619)
49. D. Frenkel, P.B. Warren, Gibbs, Boltzmann, and negative temperatures (2014). arXiv preprint [arXiv:1403.4299](https://arxiv.org/abs/1403.4299)
50. Y. Tiandho, Implication of Negative Temperature in the Inner Horizon of Reissner–Nordström Black Hole. *J. Math. Fundam. Sci.* **49**, 294 (2017)
51. S. Q. Wu, The conformal version of black hole (2004). arXiv preprint [arXiv:0411002](https://arxiv.org/abs/0411002)
52. S. Braun et al., Negative absolute temperature for motional degrees of freedom. *Science* **52**, 339 (2013)
53. D. Kastor, S. Ray, J. Traschen, Enthalpy and the mechanics of AdS black holes. *Class. Quantum Gravity* **26**, 195011 (2009)
54. M. Cvetič, G.W. Gibbons, D. Kubiznak, C.N. Pope, Black hole enthalpy and an entropy inequality for the thermodynamic volume. *Phys. Rev. D* **84**, 024037 (2011)
55. V. Moncrief, Stability of Reissner–Nordstrom black holes. *Phys. Rev. D* **10**, 1057 (1974)
56. T. Takahashi, J. Soda, *Phys. Rev. D* **80**, 104021 (2009)
57. R.-G. Cai, *Phys. Rev. D* **59**(10), 104004 (1999)
58. T. Hale, R. A. Hennigar, D. Kubiznak, Excising Cauchy horizons with nonlinear electrodynamics (2025). arXiv preprint [arXiv:2506.20802](https://arxiv.org/abs/2506.20802)
59. R. Casadio, O. Micu, D. Stojkovic, *JHEP* **05**, 096 (2015)
60. R. Casadio, A. Giugn, A. Giusti, O. Micu, *Eur. Phys. J. C* **77**, 5 (2017)